Deformation of a self-propelled domain in an excitable reaction-diffusion system

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We formulate a theory for a self-propelled domain in an excitable reaction-diffusion system in two dimensions where the domain deforms from a circular shape when the propagation velocity is increased. In the singular limit where the width of the domain boundary is infinitesimally thin, we derive a set of equations of motion for the center of gravity and two fundamental deformation modes. The deformed shapes of a steadily propagating domain are obtained. The set of time-evolution equations exhibits a bifurcation from a straight motion to a circular motion by changing the system parameters.

I. INTRODUCTION

Self-organized dynamics of domains in reaction-diffusion media have attracted much attention recently [1–3]. Formation and propagation of domains in excitable reaction-diffusion systems have been studied both theoretically and experimentally. Not only extended domain patterns such as spiral wave, but also interacting disconnected domains have been found in experiments of chemical reactions where fascinating dynamics due to propagation, collision, and self-replication of domains have been observed [4,5]. Throughout the present paper, we shall use the word "domain" for a localized isolated object in two dimensions rather than the word "pulse" which is the concept in one dimension.

Computer simulations of reaction-diffusion equations have revealed various interesting dynamics of domains [6–10]. Krischer and Mikhailov investigated numerically collisions of a pair of domains in two dimensions in an excitable reaction-diffusion system with a global coupling [10]. They have also shown that an isolated domain is deformed substantially from a circular shape when the propagating velocity is increased. It has been shown that Sierpinski gasket patterns appear in space-time plot of the pulse trajectory in one dimension in several reaction diffusion systems due to a delicate interplay between pair-annihilation and self-replication of pulses upon collision [11–13].

It has been known in excitable reaction-diffusion equations that a motionless localized domain loses its stability and begin to propagate when a system parameter exceeds the critical threshold. Reduction theories in the form of ordinary differential equations in terms of a few degrees of freedom have been developed near this bifurcation called a drift bifurcation not only in one dimension [14,15] but also in two and three dimensions [16–20]. For example, the motion of the center of gravity of an isolated spherical domain obeys in the vicinity of a supercritical drift bifurcation

$$\frac{d\boldsymbol{v}}{dt} = \gamma\boldsymbol{v} - |\boldsymbol{v}|^2\boldsymbol{v}, \tag{1}$$

where $\boldsymbol{v}$ is the velocity. The coefficient γ is negative below the threshold whereas it is positive above the threshold. The interaction between two domains can be incorporated in Eq. (1) by assuming that the characteristic length of the domain is much smaller than the distance between the domains.

Here, it is remarked that Eq. (1) for $\gamma > 0$ is a model equation for the active Brownian motion if a noise term is added [21–23].

As mentioned above, a domain is generally deformed when the propagating velocity is increased. However, to our knowledge, no analytical theory has been available for dynamics of deformable self-propelled domains. Only a stationary deformed shape of a steadily propagating domain at a constant speed has been studied by means of the interface dynamics in reaction diffusion systems [20,24,25]. (In Ref. [20], the author claims that a stable propagating domain exists in a set of two-component reaction diffusion equations whereas the authors in Refs. [24,25] study a traveling domain in the slightly different reaction diffusion equations introducing a feedback control term to stabilize the traveling domain.)

In the present paper, we shall derive the time-evolution equations for the motion of a deformed domain starting with the excitable reaction-diffusion equations studied numerically by Krischer and Mikhailov [10]. In a previous paper [26], we presented phenomenologically a set of evolution equations for $\boldsymbol{v}$ and S, the latter of which is the so-called nematic order parameter tensor [27] to represent an elliptical deformation of domain. A remarkable property of this system is that there is a bifurcation such that a straight motion of a domain becomes unstable for sufficiently small values of the relaxation rate of S and the domain undergoes a circular motion. Therefore a deformable domain exhibits two bifurcations. One is the drift bifurcation and the other is the straight-circular-motion bifurcation. Hereafter, we shall call this bifurcation a rotation bifurcation.

We shall employ the following assumptions in the derivation of the time-evolution equations for a deformable self-propelled particle. First of all, we represent the domain dynamics by means of the interface equation of motion for the domain boundary assuming that the interface width is much smaller than any other lengths of the problem [3]. The second assumption is slowness of the propagating velocity and hence weakness of deformation provided that the system is near the drift bifurcation point.

Although the present formulation is specific to the excitable reaction-diffusion equations, the set of the time-evolution equations takes a general form in terms of the vector $\boldsymbol{v}$, the second rank tensor S and the third rank tensor U. The tensor U was not considered in Ref. [26], but is neces-

sary to represent the head-tail asymmetry of a propagating domain. The tensor S represents an elliptic deformation of domain and therefore it is related to the $n = \pm 2$ mode in the Fourier series expansion of the deformation $\delta R(\phi, t) = \Sigma_n c_n(t) e^{i\phi n}$ with ϕ the angle to specify the location on the interface. The other tensor U is given in terms of the amplitudes $c_{\pm 3}$ and will be shown to be equivalent with the representation of a banana-shaped liquid crystal in two dimensions [28,29]. It is emphasized that the new tensor variable U does not enter up to the leading order in the equations for $\boldsymbol{v}$ and S and hence the previous analysis in Ref. [26]. is unaltered.

We summarize some of the related studies. First of all, it should be remarked here that there are several experiments of spontaneously deformed propagating domains. For example, an oil droplet in surfactant solutions undergoes a self-propelled motion and changes its shape depending on the propagating velocity [30,31]. The present theory would provide some insight into these experiments. Next, Shimoyama *et al.* [32] introduced a model of interacting elliptical particles, where the direction of the long axis is chosen to be an independent dynamical variable coupled to the velocity of the center of gravity. The reaction driven propulsion has been studied both experimentally and theoretically [30,31,33–37]. The active Brownian particles with a harmonic interaction between a pair of particles show a transition from a translational motion to a rotational motion by increasing the noise strength [38]. The Langevin equations for the center-of-mass position of rods and the rod orientation have been introduced to study the Brownian circular swimmer in a confined geometry [39]. Collective dynamics of self-propelled particles with an interaction between chemotactic materials and an internal degree of freedom has been investigated [40]. However, all of these theoretical studies deal with undeformable objects.

The present paper is organized as follows. In the next section (Sec. II), we start with the reaction-diffusion equations for an activator and an inhibitor with a global coupling. In Sec. III, we briefly review the interface dynamics. The representation of a deformed domain is given in Sec. IV. Apart from the center of gravity, we consider two tensor variables S and U as mentioned above. In Sec. V, the time-evolution equation for the center of gravity is derived whereas, in Sec. VI, the equations for $n = \pm 2$, and ± 3 modes are obtained to complete the set of the equations for a deformable self-propelled domain. The approximations used in the derivation of the equations and the stationary shape of a propagating domain are described in Sec. VII. Summary and discussion are given in Sec. VIII. Some of the technical details in the derivation of the equation for $\mathbf{v}$ are given in the Appendix.

II. EXCITABLE REACTION DIFFUSION SYSTEM

We start with a coupled set of reaction-diffusion equations for an activator u and an inhibitor v.

$$\tau \epsilon \frac{\partial u}{\partial t} = \epsilon^2 \nabla^2 u + f\{u, v\} - v, \tag{2}$$

$$\frac{\partial v}{\partial t} = D\nabla^2 v + u - \gamma v, \tag{3}$$

where

$$f\{u, v\} = -u + \theta(u - p'\{u, v\}), \tag{4}$$

with $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. The functional $p'\{u, v\}$ contains a global coupling as

$$p' = p + \sigma \left[\int (u + v) d\boldsymbol{r} - W \right], \tag{5}$$

where σ and W are positive constants, $0 < p < 1/2$ and the integral runs over the whole space. The constants τ and γ are positive and chosen such that the system is excitable and that a localized stable pulse (domain) solution exists. Inside the domain, the variable u is positive surrounded by the rest state where u and v vanishes asymptotically away from the domain. The parameter ϵ is a measure of the width of the domain boundary. Hereafter, we consider the limit $\epsilon \rightarrow 0$.

The set of Eqs. (2) and (3) with Eqs. (4) and (5) was introduced by Krischer and Mikhailov [10]. They investigated the collision of domains in two dimensions by computer simulations and found a reflection of a pair of colliding domains when the propagating velocity is sufficiently small. The reason why the global coupling is necessary in the system (2) and (3) is as follows. It has been known that a motionless domain is stable when τ is sufficiently large. There is a bifurcation such that below a threshold value a motionless domain becomes unstable and, when the global coupling is absent, a breathing motion occurs such that the radius of domain undergoes a periodic oscillation [41–43]. On the other hand, it is also known that the drift bifurcation exists for smaller values of τ. Therefore, when the value of τ is decreased, one generally encounters the situation that the breathing bifurcation occurs before the drift bifurcation. In order to study the intrinsic property of a propagating domain, one must exclude the breathing instability. This is achieved if one chooses a sufficiently large value of σ in the global coupling (5) so that it becomes $p' = p$ with

$$\int d\boldsymbol{r}(u + v) = W, \tag{6}$$

and $f\{u, v\}$ is no more a functional but is given by

$$f(u) = -u + \theta(u - p). \tag{7}$$

In the limit $\epsilon \rightarrow 0$ and $\sigma \rightarrow \infty$ in Eq. (5), Eq. (2) becomes

$$-u + \theta(u - p) - v = 0. \tag{8}$$

The location of the domain boundary (interface) is defined such that $u = p$ and $u > (<)p$ inside (outside) the domain. It is shown from Eq. (8) that $u + v = 1$ inside the domain whereas $u + v = 0$ outside the domain. Therefore the constraint (6) means that the volume of an excited domain is independent of time and hence the breathing motion is prohibited. It is also noted that the bifurcation from a motionless domain to a propagating domain becomes supercritical in the limit $\sigma \rightarrow \infty$ [10].

Substituting Eq. (8) into Eq. (3) yields

$$\frac{\partial v}{\partial t} = D\boldsymbol{\nabla}^2 v + \theta(u-p) - \beta v, \tag{9}$$

where $\beta = 1 + \gamma$. The equilibrium solution of a motionless circular domain with radius R_0 in two dimensions has been obtained in Ref. [42]. Noting the relation that $\theta(u-p) = \theta(R_0 - r)$, the equilibrium solution $v = \bar{v}$ is given from Eq. (9) for $0 < r < R_0$ by

$$\bar{v} = \frac{1}{\beta}[1 - R_0\kappa K_1(R_0\kappa)I_0(r\kappa)], \tag{10}$$

and for $r > R_0$ by

$$\bar{v} = \frac{R_0\kappa}{\beta}I_1(R_0\kappa)K_0(r\kappa), \tag{11}$$

where

$$\kappa = \sqrt{\frac{\beta}{D}}, \tag{12}$$

and I_n and K_n are the modified Bessel functions. The equilibrium profile of $\bar{u}$ is given by the relation $\bar{u} = \theta(R_0 - r) - \bar{v}$.

From Eqs. (10) and (11), it is clear that the inhibitor v changes gradually in space with the characteristic length κ^{-1}. On the other hand, the activator u is discontinuous at $r = R_0$ in the limit $\epsilon \to 0$ and hence there is a sharp interface. In this situation, one can derive the interface equation of motion as shown in the next section.

III. INTERFACE EQUATION OF MOTION

In order to see the spatial variation of u near $r = R_0$, one must rescale the space coordinate as $r' = r/\epsilon \sim O(1)$. In this length scale, the spatial variation of v is negligible and the value of v in Eq. (2) can be replaced by the value at the interface $v(\boldsymbol{r}, t) = w$. As a result, Eq. (2) becomes the so-called time-dependent Gizburg-Landau equation for u. The interface equation of motion for this system is well known. See, for example, Ref. [3]. Actually for a given value of w, one can readily obtain the equation of motion for an arbitrary deformed interface [42]

$$\tau V = \epsilon K + \tau c(w) + L, \tag{13}$$

where V is the normal component of the velocity directed from the inside to the outside of a domain and K is the mean curvature defined such that it is positive when the center of the curvature is outside the excited domain. The second term in Eq. (13) is the velocity for a flat interface and is related with w as [44]

$$\frac{c\tau}{\sqrt{(c\tau)^2 + 4}} = 1 - 2p - 2w. \tag{14}$$

We have used the fact that, as mentioned above, the location of the interface is defined through the condition $u(\boldsymbol{r}, t) = p$. The unknown constant w is determined by solving Eqs. (3) or (9) for a given interface configuration. The last term L is a Lagrange multiplier for the constraint of the domain area conservation (6). That is, the constant L is determined by the condition

$$\int d\omega V(\omega) = 0, \tag{15}$$

where $d\omega$ is the infinitesimal length (area in three dimensions) of the interface and the integral runs over the interface.

When the motion of the interface is slow compared with the relaxation rate of the inhibitor, one may deal with the term $\partial v / \partial t$ in Eq. (9) as a perturbation so that the asymptotic solution of Eq. (9) can be written as

$$v(\boldsymbol{r}, t) = G\theta - G^2\frac{\partial\theta}{\partial t} + G^3\frac{\partial^2\theta}{\partial t^2} - G^4\frac{\partial^3\theta}{\partial t^3} + \dots, \tag{16}$$

where G is defined through the relation

$$(-D\nabla^2 + \beta)G(\boldsymbol{r} - \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{17}$$

and the abbreviation such that $GA = \int d\boldsymbol{r}' G(\boldsymbol{r} - \boldsymbol{r}')A(\boldsymbol{r}')$ has been used. The inhibitor v given by Eq. (16) at the interface position determines the value of w. Substituting it into Eq. (13) with Eq. (14) completes the closed form of the interface equation of motion. As discussed in Ref. [18], the motion of interface is arbitrarily slow in the vicinity of the supercritical drift bifurcation where the expansion (16) is justified.

IV. DEFORMED DOMAIN

We consider a deformed domain with the center of gravity $\boldsymbol{\rho}(t)$. Its time-derivative $\dot{\boldsymbol{\rho}}$ is given by

$$\dot{\boldsymbol{\rho}} \equiv \boldsymbol{v} = \frac{1}{\Omega}\int d\omega V(\omega)\boldsymbol{R}(\omega), \tag{18}$$

where the dot means the time derivative, Ω is the area (volume in three dimensions) of the domain and

$$\boldsymbol{R}(\phi) = R(\phi)\boldsymbol{e}_r, \tag{19}$$

with the radial unit vector $\boldsymbol{e}_r$. The distance from the center of gravity to the interface is denoted by $R(\phi)$ with the angle ϕ with respect to the x axis. We assume that $R(\phi)$ is a single-valued function of ϕ for sufficiently weak deformations. The infinitesimal length $d\omega$ along the interface is related with $d\phi$ as

$$d\omega = \left|\frac{d\boldsymbol{R}}{d\phi}\right|d\phi = \sqrt{R^2 + R'^2}d\phi, \tag{20}$$

where the prime indicates the derivative with respect to ϕ. The tangential unit vector $\boldsymbol{t}$ and the unit normal $\boldsymbol{n}$ at a position on the interface are given, respectively, by

$$\boldsymbol{t} = \frac{1}{\sqrt{R^2 + R'^2}}(R'\boldsymbol{e}_r + R\boldsymbol{e}_\phi), \tag{21}$$

$$\boldsymbol{n} = \frac{1}{\sqrt{R^2 + R'^2}}(R\boldsymbol{e}_r - R'\boldsymbol{e}_\phi), \tag{22}$$

where $\boldsymbol{e}_\phi$ is the azimuthal unit vector. From these expressions, one obtains

$$K(\phi) = -\frac{R^2 + 2R'^2 - R''R}{(\sqrt{R^2 + R'^2})^3}, \tag{23}$$

$$V(\phi) = \boldsymbol{v} \cdot \boldsymbol{n} + \frac{RdR/dt}{\sqrt{R^2 + R'^2}}. \tag{24}$$

Deformations of a domain around a circular shape with radius R_0 are written as

$$R(\phi) = R_0 + \delta R(\phi, t), \tag{25}$$

where

$$\delta R(\phi, t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{in\phi}. \tag{26}$$

Note that since the translational motion of the domain has been incorporated in the variable $\boldsymbol{\rho}$, the modes $c_{\pm 1}$ should be removed from the expansion (26). The mean curvature and the normal component of the velocity are given up to the first order of the deviations, respectively, by

$$K(\phi, t) = -\frac{1}{R_0} - \frac{1}{R_0^2} \sum_{n=-\infty}^{\infty} (n^2 - 1) c_n(t) e^{in\phi}, \tag{27}$$

$$V(\phi, t) = \boldsymbol{v} \cdot \boldsymbol{n} + \sum_{n=-\infty}^{\infty} \dot{c}_n(t) e^{in\phi}, \tag{28}$$

For an isolated domain, the Fourier transform of $\theta(u-p) = \theta(R - |\boldsymbol{r} - \boldsymbol{\rho}|)$ is given by

$$\theta_q = \int_{r<R} d\boldsymbol{r} \exp(-i\boldsymbol{q} \cdot \boldsymbol{r}). \tag{29}$$

Here, the Fourier transformation is defined by

$$A(\boldsymbol{r}, t) = \int_q A_q(t) e^{i\boldsymbol{q} \cdot \boldsymbol{r}}, \tag{30}$$

$$A_q(t) = \int d\boldsymbol{r} A(\boldsymbol{r}, t) e^{-i\boldsymbol{q} \cdot \boldsymbol{r}}, \tag{31}$$

where $\int_q = \int d\boldsymbol{q}/(2\pi)^d$ with d the dimensionality of space. Substituting Eq. (25) into Eq. (29) yields up to first order of the deviations

$$\theta_q(t) = \theta_q^{(0)} + \theta_q^{(1)}, \tag{32}$$

where

$$\theta_q^{(0)} = \frac{2\pi R_0}{q} J_1(qR_0),$$

$$\theta_q^{(1)} = 2\pi R_0 \sum_n c_n(t) i^{-n} e^{in\phi} J_n(qR_0), \tag{33}$$

and $J_n(x)$ is the Bessel function of the n-th order defined by

$$\int_0^{2\pi} d\theta e^{i(n\theta \pm z \cos \theta)} = 2\pi i^{\pm n} J_n(z). \tag{34}$$

Here, we summarize the formulas for the Bessel function

$$J_{-n}(z) = (-1)^n J_n(z), \tag{35}$$

$$z^n J_{n-1}(z) = \frac{d}{dz}[z^n J_n(z)], \tag{36}$$

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z), \tag{37}$$

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z). \tag{38}$$

Since the amplitude c_n is complex, it is convenient to introduce the linear combinations of its real and imaginary parts. The modes $c_{\pm 2}$ represents an elliptical shape of domain. We introduce a second rank tensor as follows:

$$S_{11} = -S_{22} = c_2 + c_{-2},$$

$$S_{12} = S_{21} = i(c_2 - c_{-2}). \tag{39}$$

For an elliptical domain, $R(\phi)$ is represented as

$$R(\phi) = R_0 + \frac{\delta_2}{2} \cos 2(\phi - \phi_2), \tag{40}$$

where δ_2 is a positive constant and ϕ_2 is the angle between the long axis of the elliptical domain and the x axis. The tensor S can be written in terms of ϕ_2 as $S_{11} = (\delta_2/2) \cos 2\phi_2$ and $S_{12} = (\delta_2/2) \sin 2\phi_2$. This is represented in terms of the unit normal $N = (\cos \phi_2, \sin \phi_2)$ along the long axis as

$$S_{\alpha\beta} = \delta_2 \left(N_\alpha N_\beta - \frac{\delta_{\alpha\beta}}{2} N^2 \right), \tag{41}$$

which is the same as the nematic order parameter tensor in liquid crystals [27].

The modes $c_{\pm 3}$ are necessary to represent the head-tail asymmetry of a propagating domain. If the deformation $R(\phi)$ is written as

$$R(\phi) = R_0 + \delta_3 \cos 3(\phi - \phi_3), \tag{42}$$

with a positive constant δ_3, one obtains

$$T_1 = c_3 + c_{-3} = \delta_3 \cos 3\phi_3,$$

$$T_2 = i(c_3 - c_{-3}) = \delta_3 \sin 3\phi_3. \tag{43}$$

It is convenient to introduce a third-rank tensor associated with the $n = \pm 3$ modes as follows:

$$U_{\alpha\beta\gamma} = \frac{4\delta_3}{3} \sum_{m=1,2,3} N_\alpha^{(m)} N_\beta^{(m)} N_\gamma^{(m)}, \tag{44}$$

where

$$N^{(1)} = (\cos \phi_3, \sin \phi_3), \tag{45}$$

$$N^{(2)} = \left[\cos\left(\phi_3 + \frac{2\pi}{3} \right), \sin\left(\phi_3 + \frac{2\pi}{3} \right) \right], \tag{46}$$

$$N^{(3)} = \left[\cos\left(\phi_3 - \frac{2\pi}{3} \right), \sin\left(\phi_3 - \frac{2\pi}{3} \right) \right]. \qquad (47)$$

From these definitions, one obtains $U_{111} = T_1$ and $U_{222} = -T_2$. Furthermore, it is readily shown that there are relations among the components as

$$U_{111} = -U_{122} = -U_{212} = -U_{221},$$

$$U_{222} = -U_{112} = -U_{121} = -U_{211}. \qquad (48)$$

The tensor (44) is the same as the order parameter for banana (tetragonal nematic) liquid crystals in two dimensions [28,29].

In the following sections, we shall derive a coupled set of the time-evolution equations for $\boldsymbol{v}$, $c_{\pm 2}$, and $c_{\pm 3}$. However, it is impossible to derive the set of equations exactly in a general condition. As mentioned in the introduction, we employ an assumption, that is, the smallness of the propagating velocity. Since the circular shape of a domain is assumed to be stable when it is motionless, the deformation of the domain is expected to be weak near the drift bifurcation where the propagating velocity is small [18]. In this condition, the truncation of the modes up to $n = \pm 3$ is justified.

V. EQUATION OF MOTION FOR $\rho(t)$

The aim of this section is to derive the equation of motion for the center of gravity from the interface Eq. (13). When the velocity c of the flat interface is small, one may expand Eq. (14) in powers of τc. Retaining up to the third order terms, one obtains from Eq. (13)

$$2w + \frac{\tau V - \epsilon K - L}{2} - \frac{(\tau V - \epsilon K - L)^3}{16} = 1 - 2p. \qquad (49)$$

The value w of the inhibitor v at the interface may depend on the velocity $\boldsymbol{v}$ of the domain but is independent of $\boldsymbol{\rho}$ because of the translational invariance of the system. Therefore, by putting $\boldsymbol{r} = \boldsymbol{R}(\omega)$ in Eq. (16), it is given by

$$w = w_{(0)} + w_{(1)} + w_{(2)} + w_{(3)}, \qquad (50)$$

where

$$w_{(0)} = \int_q G_q \theta_q e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (51)$$

$$w_{(1)} = w_{(11)} + w_{(12)} = i \int_q (\boldsymbol{v}\cdot\boldsymbol{q}) G_q^2 \theta_q e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}$$
$$- \int_q G_q^2 \left(\frac{\partial \theta_q}{\partial t} \right) e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (52)$$

$$w_{(2)} = w_{(21)} + w_{(22)} + w_{(23)}$$
$$= -i \int_q (\dot{\boldsymbol{v}}\cdot\boldsymbol{q}) G_q^3 \theta_q e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)} - \int_q (\boldsymbol{v}\cdot\boldsymbol{q})^2 G_q^3 \theta_q e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}$$
$$+ \int_q G_q^3 \left(\frac{\partial^2 \theta_q}{\partial t^2} \right) e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (53)$$

$$w_{(3)} = -i \int_q (\boldsymbol{v}\cdot\boldsymbol{q})^3 G_q^4 \theta_q e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (54)$$

with

$$G_q = \frac{1}{D(q^2 + \kappa^2)}. \qquad (55)$$

We have dropped out terms having a factor $\boldsymbol{v}\dot{\theta}$ in Eq. (53) and $\boldsymbol{v}\boldsymbol{v}\dot{\theta}$ and the second derivatives in Eq. (54).

In order to obtain the equation for $\boldsymbol{\rho}$, we operate $(1/\Omega)\int d\omega \boldsymbol{R}(\omega)$ to Eq. (49). As mentioned at the end of the preceding section, the basic approximation is the expansion in terms of $\boldsymbol{v}$ and c_n ignoring the higher order time derivatives. The validity of these approximations will be discussed in detail in Sec. VII.

Since the Lagrange multiplier L is independent of the angle ϕ, one has $\int d\omega \boldsymbol{R}(\omega) L = \int d\omega \boldsymbol{R}(\omega) L^3 = \int d\omega \boldsymbol{R}(\omega) V^2 L = 0$ for a circular domain. Furthermore, it will be shown later in Sec. IV [just after Eq. (81)] that $L \sim O(\boldsymbol{v}^2)$, one may ignore $\int d\omega \boldsymbol{R}(\omega) V L^2$ up to the third order of $\boldsymbol{v}$. It is also noted that $\int d\omega \boldsymbol{R}(\omega) K(\omega) = 0$ up to the first order of deformations since the modes $c_{\pm 1}$ are excluded in Eq. (26). The term $\epsilon V K$ and the higher order terms of ϵ are ignored. As a result, $\epsilon K + L$ in the third term on the left hand side of Eq. (49) can be dropped out in the derivation of equation for $\boldsymbol{\rho}$.

The zero-th order terms in Eq. (49) that represent a motionless circular domain are given by

$$-\frac{\epsilon}{2R_0} + 1 - 2p - 2\int_q G_q \theta_q^{(0)} e^{i\boldsymbol{q}\cdot\boldsymbol{R}^{(0)}(\omega)} = 0, \qquad (56)$$

with $\boldsymbol{R}^{(0)} = R^{(0)} \boldsymbol{e}_r$. The Lagrange multiplier L can be omitted at this order since it is absorbed into the constant p. Equation (56) gives us the equilibrium radius R_0 of the circular domain as a function, e.g., of the parameter p [42]. Since the radius has been fixed by the constraint (6), this implies that the parameter p is not independent but should be related to W in Eq. (6).

From the higher order terms in Eq. (49), one obtains in two dimensions

$$2w + \frac{\tau}{2}\boldsymbol{v} - \frac{3\tau^3}{64}\boldsymbol{v}|\boldsymbol{v}|^2 = 0, \qquad (57)$$

where

$$w = w_{(11)} + w_{(21)} + w_{(3)}, \qquad (58)$$

with

$$w_{(11)} = \frac{1}{\Omega}\int d\omega \int_q (\boldsymbol{v}\cdot\boldsymbol{q}) G_q^2 \theta_q \frac{\partial}{\partial \boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (59)$$

$$w_{(21)} = -\frac{1}{\Omega}\int d\omega \int_q (\dot{\boldsymbol{v}}\cdot\boldsymbol{q}) G_q^3 \theta_q \frac{\partial}{\partial \boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{R}(\omega)}, \qquad (60)$$

$$w_{(3)} = -\frac{1}{\Omega} \int d\omega \int_q (\boldsymbol{v} \cdot \boldsymbol{q})^3 G_q^4 \theta_q \frac{\partial}{\partial \boldsymbol{q}} e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)}. \quad (61)$$

These have been evaluated for a circular domain in two dimensions previously [18]. Substituting those results into Eq. (57) gives us

$$m\dot{\boldsymbol{v}} + \frac{1}{2}(\tau - \tau_c)\boldsymbol{v} + g\boldsymbol{v}|\boldsymbol{v}|^2 = -2\delta w_{(11)}, \quad (62)$$

where

$$m = 2R_0 \int_0^\infty dq q G_q^3 J_1(qR_0)^2, \quad (63)$$

$$\tau_c = 4R_0 \int_0^\infty dq q G_q^2 J_1(qR_0)^2, \quad (64)$$

$$g = \frac{3R_0}{2} \int_0^\infty dq q^3 G_q^4 J_1(qR_0)^2 - \frac{3\tau^3}{64}. \quad (65)$$

The left hand side of Eq. (62) has been obtained in Ref. [18]. The right hand side is defined as $\delta w_{(11)} = w_{(11)} - w_{(11)}^{(0)}$ and vanishes identically for a circular domain where $w_{(11)}^{(0)}$ is given by (59) with $\theta_q = \theta_q^{(0)}$ and $\boldsymbol{R}(\omega) = \boldsymbol{R}^{(0)}(\omega)$. The coefficient m is positive and g is shown to be positive for $\tau \approx \tau_c$. Therefore, Eq. (62) indicates a bifurcation (drift bifurcation) such that a motionless circular domain $\boldsymbol{v} = 0$ is stable for $\tau > \tau_c$ whereas it looses stability for $\tau < \tau_c$ and undergoes propagation at the velocity $|\boldsymbol{v}|^2 = (\tau_c - \tau)/(2g)$.

The evaluation of the right hand side $\delta w_{(11)}$ of Eq. (62) up to the order of $O(\boldsymbol{v}c_n)$ is given in the Appendix. By using the expressions of $H_{\alpha\beta}^{(n)}$ with $n=1$ and 2 obtained there, Eq. (62) is finally written as

$$m\dot{v}_\alpha + \frac{1}{2}(\tau - \tau_c)v_\alpha + g v_\alpha |\boldsymbol{v}|^2 = -a v_\beta S_{\beta\alpha}, \quad (66)$$

where the repeated indices imply summation and

$$a = 2(a_1 + a_2), \quad (67)$$

with a_1 and a_2 given, respectively, by Eqs. (A5) and (A10). The tensor $S_{\beta\alpha}$ has been defined in Eq. (41) with Eq. (39). The coefficient a is evaluated numerically and is displayed as a function of $\hat{R}_0 = R_0\kappa$ in Fig. 1. It is noted that the coefficient a is negative for $0 < \hat{R}_0 < \infty$. When $\hat{R}_0 \ll 1$, one has an analytical expression $\hat{a} \equiv aD^2/R_0^2 = -(3/4)|\ln \hat{R}_0|$. The numerical result in Fig. 1 is consistent with this.

VI. EQUATIONS OF MOTION FOR $c_{\pm 2}(t)$ AND $c_{\pm 3}(t)$

In order to derive the time-evolution equations for $c_{\pm 2}(t)$ and $c_{\pm 3}(t)$, we start with the interface equation of motion (13). Since it will be shown in Sec. VII that $c_{\pm 2}(t) \sim O(\boldsymbol{v}^2)$ and $c_{\pm 3}(t) \sim O(\boldsymbol{v}^3)$, the coupling such as $c_n c_m$ is negligible for slow dynamics of a domain. Therefore, we may consider only the linear order of the deviation δR. (This takes account of the coupling between c_n and $\boldsymbol{v}$.) Linearizing Eq. (13) with respect to the deformation, one obtains
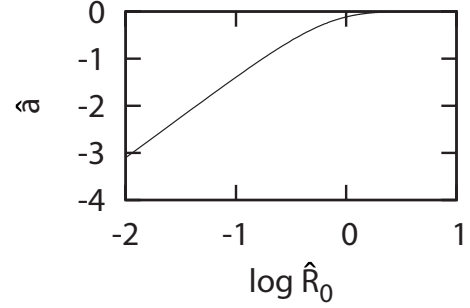

FIG. 1. The coefficient $\hat{a} = aD^2/R_0^2$ as a function of $\hat{R}_0 = \kappa R_0$. Here and in Figs. 2 and 3 below, the base of the logarithmic function is 10.

$$\tau \frac{\partial \delta R}{\partial t} = \frac{\epsilon}{R_0^2} \left(\frac{\partial^2 \delta R}{\partial \phi^2} + \delta R \right) - 4\delta w + L, \quad (68)$$

where $\delta w = w - w_0$ with w_0 the value of w for a circular domain and we have used the fact that $\tau d c(w_0)/dw_0 = -4$ as obtained from Eq. (14). The last term L is necessary in order to eliminate any zero modes which might arise from δw.

First, we consider the $n = \pm 2$ modes. From Eqs. (51)–(53), one notes that there are three terms in δw, which produce $n = \pm 2$ modes;

$$\delta w = \delta w_0 + \delta w_1 + \delta w_2, \quad (69)$$

where

$$\delta w_0 = \int_q G_q \theta_q e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)} - \int_q G_q \theta_q^{(0)} e^{i\boldsymbol{q} \cdot \boldsymbol{R}^{(0)}(\omega)}, \quad (70)$$

$$\delta w_1 = \delta w_{11} + \delta w_{12} = i \int_q (\boldsymbol{q} \cdot \boldsymbol{v}) G_q^2 \theta_q e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)}$$
$$- \int_q G_q^2 \frac{\partial \theta_q}{\partial t} e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)}, \quad (71)$$

$$\delta w_2 = -\int_q (\boldsymbol{q} \cdot \boldsymbol{v})^2 G_q^3 \theta_q e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)}. \quad (72)$$

Here, we have ignored the term with $\partial^2 \theta_q / \partial t^2$ which causes $\ddot{c}_n$ and are higher order in the final equation of motion. By using Eq. (33), these are readily evaluated as

$$\delta w_0 = -\sum_n D_n c_n e^{in\phi}, \quad (73)$$

$$\delta w_{11} = -\frac{1}{2}(v_1 - iv_2) \sum_n A_n c_n e^{i(n+1)\phi}$$
$$- \frac{1}{2}(v_1 + iv_2) \sum_n B_n c_n e^{i(n-1)\phi}, \quad (74)$$

$$\delta w_{12} = -\sum_n E_n \dot{c}_n e^{in\phi}, \quad (75)$$

$$\delta w_2 = -\frac{G_0}{2}\boldsymbol{v}^2 + \frac{G_1}{2}\left[\left(\frac{1}{2}(v_1^2 - v_2^2) - iv_1v_2\right)e^{2i\phi} + \left(\frac{1}{2}(v_1^2 - v_2^2)\right.\right.$$

$$\left.\left. + iv_1v_2\right)e^{-2i\phi}\right], \tag{76}$$

where

$$A_n = R_0\int_0^\infty dq q^2 G_q^2\left[J_n(qR_0)J_{n+1}(qR_0)\right.$$

$$\left. + J_1(qR_0)\frac{\partial}{\partial qR_0}J_1(qR_0)\right], \tag{77}$$

$$B_n = R_0\int_0^\infty dq q^2 G_q^2\left[-J_n(qR_0)J_{n-1}(qR_0)\right.$$

$$\left. + J_1(qR_0)\frac{\partial}{\partial qR_0}J_1(qR_0)\right], \tag{78}$$

$$D_n = R_0\int_0^\infty dq q G_q[J_1(qR_0)^2 - J_n(qR_0)^2], \tag{79}$$

$$E_n = R_0\int_0^\infty dq q G_q^2 J_n(qR_0)^2, \tag{80}$$

$$G_\ell = R_0\int_0^\infty dq q^2 G_q^3 J_\ell(qR_0)J_{\ell+1}(qR_0). \tag{81}$$

Note the relation that $A_{-n} = B_n$. It is also noted from Eq. (76) that δw_2 contains the $n = 0$ mode proportional to $\boldsymbol{v}^2$, which should be absorbed into the Lagrange multiplier L in Eq. (68). From the $n = \pm 2$ modes, one obtains

$$\delta w_2^{n=2} + \delta w_2^{n=-2} = \frac{G_1}{2}(v_1^2 - v_2^2), \tag{82}$$

$$i(\delta w_2^{n=2} - \delta w_2^{n=-2}) = G_1 v_1 v_2. \tag{83}$$

After extracting the $n = \pm 2$ terms from Eq. (68), equation for the tensor S defined by Eq. (41) is given by

$$\Gamma_2\frac{dS_{\alpha\beta}}{dt} = -K_2 S_{\alpha\beta} + b\left[v_\alpha v_\beta - \frac{\delta_{\alpha\beta}}{2}\boldsymbol{v}^2\right] + b_1 U_{\alpha\beta\gamma}v_\gamma, \tag{84}$$

where $\Gamma_2 = \tau - 4E_2$, $b = -4G_1$, $K_2 = (3\epsilon/R_0^2) - 4D_2$ and $b_1 = 2B_3$. It is noted that the coefficient b is negative. In the vicinity of the bifurcation $\tau \sim \tau_c$ with τ_c given by Eq. (64), the coefficient Γ_2 is positive. The coefficient K_2 becomes negative for sufficiently large values of R_0 indicating an instability of a motionless circular domain [42]. The coefficient b_1 is plotted as a function of $\hat{R}_0$ in Fig. 2. It is positive for $\hat{R}_0 < 0.8$ and negative for $\hat{R}_0 > 0.8$.

By putting all the $n = \pm 3$ terms together, the time-evolution equations for $c_{\pm 3}$ or T_i ($i = 1$ and 2) defined by Eq. (43) take the following form:
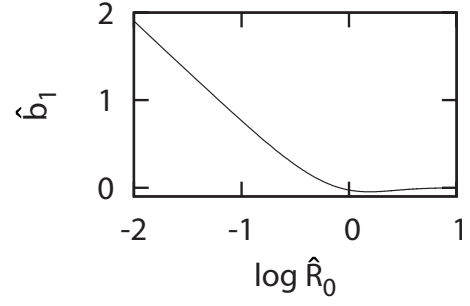

FIG. 2. The coefficient $\hat{b}_1 = D^2 b_1/R_0^2$ as a function of $\hat{R}_0 = \kappa R_0$.

$$\Gamma_3\frac{dT_1}{dt} = -K_3 T_1 + d_1 v_1(v_1^2 - 3v_2^2) + d_2(v_1 S_{11} - v_2 S_{12}), \tag{85}$$

$$\Gamma_3\frac{dT_2}{dt} = -K_3 T_2 - d_1 v_2(v_2^2 - 3v_1^2) - d_2(v_2 S_{22} - v_1 S_{21}), \tag{86}$$

where $\Gamma_3 = \tau - 4E_3$ and $K_3 = 8\epsilon/R_0^2 - 4D_3$ and the relation $S_{11} = -S_{22}$ has been used. The second term in Eqs. (85) and (86) arises from $w_{(3)}$ given by Eq. (54), whereas the third term in Eqs. (85) and (86) comes from $w_{(11)}$ given by Eq. (52), and the coefficients d_1 and d_2 are given, respectively, by

$$d_1 = R_0\int_0^\infty dq q^3 G_q^4 J_1(qR_0)J_3(qR_0), \tag{87}$$

$$d_2 = 2R_0\int_0^\infty dq q^2 G_q^2\left[J_2(qR_0)J_3(qR_0)\right.$$

$$\left. + J_1(qR_0)\frac{\partial}{\partial qR_0}J_1(qR_0)\right]. \tag{88}$$

The coefficient d_2 is displayed as a function of $\hat{R}_0$ in Fig. 3. Both d_1 and d_2 are positive.

Equations (85) and (86) can be represented in terms of the third-rank tensor $U_{\alpha\beta\gamma}$ defined by Eq. (44) as
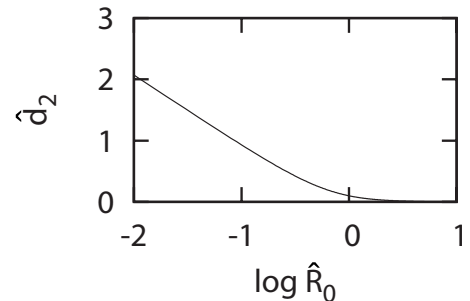

FIG. 3. The coefficient $\hat{d}_2 = d_2 D^2/R_0^2$ as a function of $\hat{R}_0 = \kappa R_0$.

$$\Gamma_3 \frac{dU_{\alpha\beta\gamma}}{dt} = -K_3 U_{\alpha\beta\gamma} + 4d_1\left[v_\alpha v_\beta v_\gamma - \frac{v_\eta v_\eta}{4}(\delta_{\alpha\beta}v_\gamma + \delta_{\beta\gamma}v_\alpha \right.$$
$$\left. + \delta_{\gamma\alpha}v_\beta) \right] + \frac{2d_2}{3}\left[S_{\alpha\beta}v_\gamma + S_{\beta\gamma}v_\alpha + S_{\gamma\alpha}v_\beta \right.$$
$$\left. - \frac{v_\eta}{2}(\delta_{\alpha\beta}S_{\gamma\eta} + \delta_{\beta\gamma}S_{\alpha\eta} + \delta_{\gamma\alpha}S_{\beta\eta}) \right]. \quad (89)$$

Other terms such as $U\boldsymbol{v}\boldsymbol{v}$ are higher order as shown in the next section.

VII. ORDER ESTIMATION AND STATIONARY SOLUTIONS

In this section, we shall discuss how to justify the approximations employed in the derivations of the time-evolution equations Eqs. (66), (84), and (89). As emphasized, the basic assumption is that we are concerned with the domain dynamics near the supercritical drift bifurcation $\tau = \tau_c$. Therefore, the smallness parameter is $\delta \equiv \tau_c - \tau$. It is readily shown from Eq. (66) that all the terms are of the order of $\delta^{3/2}$ since $\boldsymbol{v} \sim O(\delta^{1/2})$ and time is scaled as $\hat{t} = t\delta$ and $S \sim O(\delta)$ as can be seen from the first and second terms in Eq. (84). Similarly, one notes from Eq. (89) that $U \sim O(\delta^{3/2})$.

The above order estimation tells us that the terms ignored in Eq. (66) are indeed higher order in δ. For example, $d^2\boldsymbol{v}/dt^2 \sim O(\delta^{5/2})$ and $SS\boldsymbol{v} \sim O(\delta^{5/2})$. On the other hand, the first and second terms in Eq. (84) are of the order of $O(\delta)$ whereas $dS/dt \sim O(\delta^2)$ and $U\boldsymbol{v} \sim O(\delta^2)$ and all other terms ignored are higher order. Equation (89) has also a similar property. Therefore, up to the leading order, the time derivative in S and U should be ignored. This is not surprising since, for $\delta \to 0$, the center of gravity is a slow variable but the deformations around a circular shape are not necessarily slow and might be eliminated adiabatically. This fact does not cause any difficulties in the study of the stationary shape of a steadily propagation domain.

It is more convenient to employ the following representation:

$$v_1 = v\cos\phi, \quad v_2 = v\sin\phi, \quad (90)$$

$$S_{11} = \frac{1}{2}s\cos 2\theta, \quad S_{12} = \frac{1}{2}s\sin 2\theta, \quad (91)$$

$$U_{111} = c_3 + c_{-3}, \quad U_{222} = -i(c_3 - c_{-3}), \quad (92)$$

$$c_3 = \frac{\delta_3}{2}e^{-3i\phi_3} = \frac{z}{4}e^{-3i\varphi}, \quad (93)$$

where v should not be confused with the inhibitor v in Eq. (3). Substituting these into Eqs. (66), (84), and (89) yields

$$\frac{d}{dt}v = v(\gamma - v^2) - \frac{1}{2}a'sv\cos 2(\theta - \phi), \quad (94)$$

$$\frac{d}{dt}\phi = -\frac{1}{2}a's\sin 2(\theta - \phi), \quad (95)$$

$$\frac{d}{dt}s = -\kappa s + b'v^2\cos 2(\theta - \phi) + b_1'zv\cos(2\theta + \phi - 3\varphi), \quad (96)$$

$$\frac{d}{dt}\theta = -\frac{b'v^2}{2s}\sin 2(\theta - \phi), \quad (97)$$

$$\frac{d}{dt}z = -Kz + d_1'v^3\cos 3(\varphi - \phi) + d_2'sv\cos(3\varphi - 2\theta - \phi), \quad (98)$$

$$\frac{d}{dt}\varphi = -\frac{d_1'}{3z}v^3\sin 3(\varphi - \phi) - \frac{d_2'}{3z}sv\sin(3\varphi - 2\theta - \phi), \quad (99)$$

where $\gamma = (\tau_c - \tau)/(2m)$, $a' = a/m$, $\kappa = K_2/\Gamma_2$, $b' = b/\Gamma_2$, $b_1' = b_1/\Gamma_2$, $K = K_3/\Gamma_3$, $d_1' = 2d_1/\Gamma_3$, and $d_2' = d_2/\Gamma_3$ and we have put $g/m = 1$ without loss of generality. In what follows, we drop the prime in a', b', b_1', d_1' and d_2'.

Hereafter, we shall discuss the properties of the time-evolution Eqs. (66), (84), and (89) from more general point of view. In fact, all the terms are derived by considering the possible couplings which can be generated from the first rank tensor (vector) $\boldsymbol{v}$, the second rank tensor S and the third rank tensor U not necessarily restricted to the reaction-diffusion equations. Therefore, if the coefficient $\Gamma_2 \sim O(\delta^{-1})$ (and $\Gamma_3 \sim O(1)$), one has to retain the dS/dt term in Γ_2 whereas the $U\boldsymbol{v}$ term of the order of $O(\delta^2)$ can be ignored. In this case, Eqs. (94)–(97) are closed. In the previous paper, we have shown that there are two stable stationary solutions depending on the parameters [26]. For simplicity, we restrict ourselves to $ab > 0$. Here, we consider a straight motion propagating, without loss of generality, along the x axis, i.e., $\phi = 0$. When b is positive, $\theta = 0$, and the steady value of the amplitudes is given by $v_0^2 = \gamma/(1 + B)$ and $s_0 = bv_0^2/\kappa$ where $B \equiv ab/(2\kappa)$. whereas, when b is negative, the solution is given by $\theta = \pi/2$, $v_0^2 = \gamma/(1 + B)$ and $s_0 = -bv_0^2/\kappa$.

As shown in the preceding section, the both constants a and b are negative in the reaction diffusion equations studied in this paper. In this case, the long axis of an elliptical domain is perpendicular to the velocity vector. Substituting the stationary solutions given above into Eqs. (98) and (99), one obtains the time-independent solution

$$Kz_0 = d_1v_0^3\cos 3\varphi_0 + d_2s_0v_0\cos(3\varphi_0 - \pi), \quad (100)$$

$$0 = d_1v_0^3\sin 3\varphi_0 + d_2s_0v_0\sin(3\varphi_0 - \pi). \quad (101)$$

Since z_0 should be positive, we find the following condition irrespective of the sign of b:

$$d_1 + \frac{b}{\kappa}d_2 > 0 \Rightarrow \cos 3\varphi_0 = 0 \Rightarrow \varphi_0 = 0, \quad (102)$$

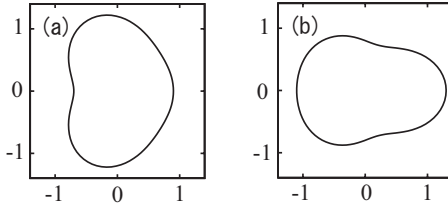$$d_1 + \frac{b}{\kappa}d_2 < 0 \Rightarrow \cos 3\varphi_0 = -1 \Rightarrow \varphi_0 = \pi, \quad (103)$$

and

FIG. 4. (a) Stationary shape of a domain with $R_0=1$ traveling along the x axis ($\phi=0$) for $\varphi_0=0$ and (a) $\theta=\pi/2$, and (b) $\theta=0$.

$$z_0 = \left| d_1 + \frac{b}{\kappa} d_2 \right| \frac{v_0^3}{K}. \tag{104}$$

The stationary shape of a domain propagating along the x axis ($\phi=0$) is shown in Fig. 4(a) for $\varphi_0=0$. It should be noted that in the reaction diffusion system studied in this paper, the constants d_1 and d_2 are positive whereas b is negative. When κ is large, the straight motion is stable and the inequality (102) always holds. Therefore, we have the deformed domain shown in Fig. 4(a). We emphasize that this is consistent with the numerical simulations in Ref. [10]. On the other hand, when b is positive (the constant a is also assumed to be positive), the long axis of an elliptical domain is parallel to the velocity vector [26]. In this case, the stationary domain shape is displayed in Fig. 4(b) for $\varphi_0=0$. If the condition (103) holds, i.e., $\varphi_0=\pi$, the shape of domain is given by the mirror symmetry with respect to the vertical axis of those in Fig. 4.

In our previous paper [26], we have shown that there is a bifurcation such that the straight motion becomes unstable and the domain undergoes a circular motion. This rotation bifurcation occurs at

$$\gamma = \gamma_c = \frac{\kappa^2}{ab} + \frac{\kappa}{2}, \tag{105}$$

such that the stationary straight motion becomes unstable for $\gamma \gtrsim \gamma_c$. Assuming a circular motion appears for $\gamma \gtrsim \gamma_c$, we put $v=v_r$, $s=s_r$, $\theta=\omega t + \zeta/2$ and $\phi=\omega t$. Substituting these into Eqs. (94)–(97), one obtains after some algebra $v_r^2=\gamma - \kappa/2$, $s_r^2=bv_r^2/a$, $\cos\zeta = \kappa/as_r$ and $\omega^2=(ab/4)(v_r^2-v_c^2)$ where $v_c=\kappa/(ab)^{1/2}$. It has been shown that the frequency ω continuously increases from zero at $\gamma=\gamma_c$. We have carried out the stability of the straight motion analytically and that of the circular motion numerically and have found that when the straight motion is stable, the circular motion is unstable and vise versa. That is, there is no parameter regime where these two motions coexist [26]. The phase diagram for the drift bifurcation and the rotation bifurcation is shown in Fig. 5. This is obtained from the equations for $\boldsymbol{v}$ and S. A more complete phase diagram including the dynamics of U will be investigated separately.

The stationary shape of a circular motion can also be obtained without any difficulty. There are two types of circular motion; clockwise and anticlockwise rotation. We consider only the anticlockwise rotation putting $\varphi=\omega t + \delta/3$. Then, the set of Eqs. (98) and (99) becomes

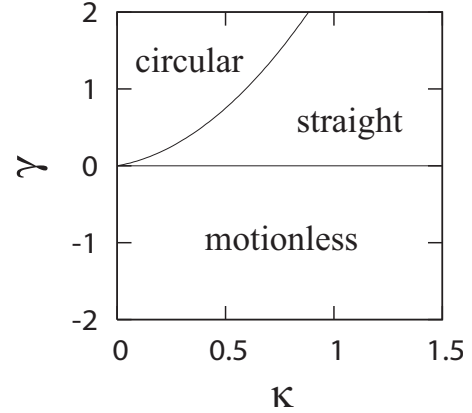$$Kz_r = d_1v_r^3 \cos\delta + d_2s_rv_r \cos(\delta-\zeta), \tag{106}$$

FIG. 5. Phase diagram on the $\gamma-\kappa$ plane for $a=-1$ and $b=-0.5$.

$$3z_r\omega = -d_1v_r^3 \sin\delta - d_2s_rv_r \sin(\delta-\zeta). \tag{107}$$

From these two equations, the stationary solution is given by

$$\frac{\cos\delta}{z_r} = \frac{(d_1v_r^2+\kappa d_3)K-6\omega^2 d_3}{d_1^2v_r^5+d_3v_r^3(2\kappa d_1+bd_2)}, \tag{108}$$

$$\frac{\sin\delta}{z_r} = -\frac{2\omega d_3K+3\omega(d_1v_r^2+\kappa d_3)}{d_1^2v_r^5+d_3v_r^3(2\kappa d_1+bd_2)}, \tag{109}$$

where $d_3=d_2/a$ and

$$\tan\delta = -\frac{2\omega d_3K+3\omega(d_1v_r^2+\kappa d_3)}{(d_1v_r^2+\kappa d_3)K-6\omega^2 d_3}. \tag{110}$$

From Eqs. (108) and (109), one has

$$z_r^2 = \frac{d_1^2v_r^6+d_3v_r^4(2\kappa d_1+bd_2)}{K^2+9\omega^2}. \tag{111}$$

A sequence of the snapshots of the domain shape in the stationary circular motion is given in Fig. 6 where the domain undergoes a clockwise motion. The replacements $\omega \rightarrow -\omega$,
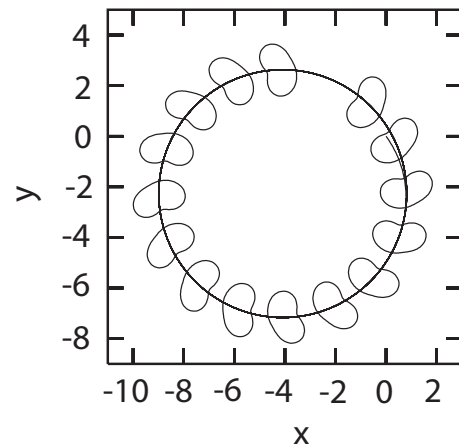

FIG. 6. Snapshots of a domain in a clockwise circular motion for $a=-1.0$, $b=-0.5$, $b_1=0$, $d_1=0.25$, $d_2=0.15$, $\gamma=1.0$, $\kappa=0.5$, and $K=0.3$. The solid circle indicates the trajectory of the center of gravity.

$\zeta \rightarrow -\zeta$, and $\delta \rightarrow -\delta$ give us the solution for a clockwise rotation.

VIII. SUMMARY AND DISCUSSION

In this paper, we have studied the domain dynamics in an excitable reaction-diffusion system with the global coupling. The set of time-evolution equations has been derived in the singular limit that the interface width is infinitesimal. The main results are Eqs. (66), (84), and (89). The coefficient a in Eq. (66) and b in Eq. (84) are negative whereas the coefficients d_1 and d_2 in Eq. (89) are positive. Therefore, the deformed shape in the steady state takes the form shown in Fig. 4(a). We emphasize that this result is consistent with the computer simulations of Eqs. (2) and (3) with Eqs. (4) and (5) [10].

We do not claim that the present theory is directly applicable to the experiments where a self-propelled domain is deformed into a banana shape as the velocity is increased [30,31]. In fact, since these phenomena are observed in an aqueous solution, the hydrodynamic effect might be important. The interaction between the domain and the solid substrate is also expected to be relevant [30,34]. What we emphasize is that, if these interactions specific to each system can be eliminated, e.g., by a reduction theory, the motion of a domain should be described by Eqs. (66), (84), and (89). The specific interactions may influence the coefficients such as a, b, d_1, and d_2 but the form of the equations is general and independent of the details of the system as long as the effective interaction along the interface is not extremely nonlocal. This property is useful, for example, to look for the possibility of the rotation bifurcation shown in the phase diagram in Fig. 5. In fact, our prediction of the rotation bifurcation (which seems not to have been observed in real experiments so far) is that it is more likely to occur when κ is small, in other words, when the domain is soft against deformations.

The present theory is restricted to two dimensions. Extension to three dimensions is possible if one expands deformations around a spherical shape in terms of spherical harmonics. Representation of the set of equations by introducing some tensor variables as in the present theory is also convenient. However, it is expected that a propagating domain in three dimensions are elongated not only to a rod shape or a cone shape [corresponding to the form in Fig. 4(b)] parallel to the propagating direction but also to a jellyfish-like shape [corresponding to the form in Fig. 4(a)] when the elongation is perpendicular to the propagating direction. Therefore, the direct use of the same tensors $S_{\alpha\beta}$ for the nematic order parameter and $U_{\alpha\beta\gamma}$ for banana-shaped liquid crystals seems not to be possible.

Apart from the generalization to three dimensions, there are several important open problems. First of all, the present theory should be generalized to a collision process of a pair of deformable self-propelled domains. If the interaction is repulsive, the motion of the two domains becomes slow as approaching to each other and hence those shapes are expected to change as shown in the present theory. However, the existence of another domain itself tends to deform generally the domain, which should be incorporated into our theory. Next, the dynamics of domains undergoing the circular motion will be investigated in detail when the global orientational coupling is imposed. Since a clockwise motion and a counterclockwise motion coexists, the orientational ordering is interrupted so that complex dynamics, such as synchronization, localization and chaotic motions appear depending on the parameters [26]. Numerical simulations and a theoretical study reducing to nonlinear coupled oscillators are now being carried out [45]. Third, the helical motion in three dimensions, which is an extension of the circular motion in two dimensions, must be investigated both numerically and analytically. Finally, the stochastic dynamics of deformable self-propelled domains will be developed by adding random noise terms in the equations derived here. As a related study, quite recently, Matsuo *et al.* [46] introduced and investigated a dynamical model for the motion of amoeboid cell to analyze the experimental results obtained. We hope to publish these investigations somewhere in the near future.

APPENDIX

In this appendix, we derive the explicit expression of $\delta w_{(11)}$ in Eq. (62). Up to the order of $O(\boldsymbol{v}c_n)$, $\delta w_{(11)}$ is given by

$$\delta w_{(11)} \equiv \delta w_{(11)}^{(1)} + \delta w_{(11)}^{(2)}$$

$$= \frac{1}{\Omega} \int d\omega \int_{\boldsymbol{q}} (\boldsymbol{v} \cdot \boldsymbol{q}) G_q^2 \theta_q^{(1)} \frac{\partial}{\partial \boldsymbol{q}} e^{i\boldsymbol{q} \cdot \boldsymbol{R}^{(0)}(\omega)}$$

$$+ \frac{1}{\Omega} \int d\omega \int_{\boldsymbol{q}} (\boldsymbol{v} \cdot \boldsymbol{q}) G_q^2 \theta_q^{(0)} \frac{\partial}{\partial \boldsymbol{q}} e^{i\boldsymbol{q} \cdot \boldsymbol{R}^{(0)}(\omega)}. \quad \text{(A1)}$$

The following formula is useful to evaluate $\delta w_{(11)}$:

$$\frac{1}{\Omega} \int d\omega e^{i\boldsymbol{q} \cdot \boldsymbol{R}(\omega)} = \frac{2}{R_0} J_0(qR_0) + \frac{q}{\pi R_0} \sum_n c_n(t) e^{in\theta} 2\pi i^n J_{n-1}(qR_0)$$

$$+ \frac{1}{\pi R_0^2} \sum_n c_n(t) e^{in\theta} 2\pi i^n (1-n) J_n(qR_0),$$

$$\text{(A2)}$$

where θ is the angle between $\boldsymbol{q}$ and the x axis, i.e., $\boldsymbol{q} = q(\cos \theta, \sin \theta)$. This formula is valid up to the first order of the deformation.

Now, we calculate each term in Eq. (A1). The first term is written as

$$\delta w_{(11,\beta)}^{(1)} = v_\alpha H_{\alpha\beta}^{(1)}, \quad \text{(A3)}$$

where the repeated indices imply the summation and

$$H_{11}^{(1)} = -4\pi R_0 \sum_n c_n(t) i^{-n} \int_q \frac{q_x^2}{q} G_q^2 e^{in\theta} J_1(qR_0) J_n(qR_0)$$

$$= a_1(c_2 + c_{-2}), \tag{A4}$$

with

$$a_1 = \frac{R_0}{2} \int_0^\infty dq q^2 G_q^2 J_1(qR_0) J_2(qR_0). \tag{A5}$$

Similarly, one obtains

$$H_{12}^{(1)} = -4\pi R_0 \sum_n c_n(t) i^{-n} \int_q \frac{q_x q_y}{q} G_q^2 e^{in\theta} J_1(qR_0) J_n(qR_0)$$

$$= a_1(c_2 - c_{-2})i. \tag{A6}$$

together with the relations $H_{22}^{(1)} = -H_{11}^{(1)}$ and $H_{12}^{(1)} = H_{21}^{(1)}$. It should be noted that because of the factors q_x^2 in Eq. (A4) and $q_x q_y$ in Eq. (A6), only the components $n = \pm 2$ contribute to $H_{11}^{(1)}$ and $H_{12}^{(1)}$.

The second term in Eq. (A1) can be written by using the formula (A2) as

$$\delta w_{(11,\beta)}^{(2)} = v_\alpha H_{\alpha\beta}^{(2)}, \tag{A7}$$

where

$$H_{\alpha\beta}^{(2)} = \frac{2}{R_0} \sum_n c_n(t) i^n \int_q q_\alpha G_q^2 \theta_q^{(0)} \frac{\partial}{\partial q_\beta} e^{in\theta}$$

$$\times \left[qJ_{n-1}(qR_0) + \frac{1-n}{R_0} J_n(qR_0) \right]$$

$$= -\frac{2}{R_0} \sum_n c_n(t) i^n \int_q e^{in\theta} \left[qJ_{n-1}(qR_0) \right.$$

$$\left. + \frac{1-n}{R_0} J_n(qR_0) \right] \frac{q_\alpha q_\beta}{q} \frac{\partial}{\partial q} (G_q^2 \theta_q^{(0)}). \tag{A8}$$

In the derivation of the second equality, we have used the fact $c_0 = 0$ which comes from the domain area conservation. Each component of Eq. (A8) is readily obtained as

$$H_{11}^{(2)} = a_2(c_2 + c_{-2}),$$

$$H_{12}^{(2)} = a_2(c_2 - c_{-2})i, \tag{A9}$$

where $H_{22}^{(2)} = -H_{11}^{(2)}$ and $H_{12}^{(2)} = H_{21}^{(2)}$ and

$$a_2 = \frac{1}{4\pi R_0} \int_0^\infty dq q^2 \left[qJ_1(qR_0) - \frac{1}{R_0} J_2(qR_0) \right] \frac{\partial}{\partial q} (G_q^2 \theta_q^{(0)}). \tag{A10}$$

By using these results, one obtains Eq. (66).

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